

# On fractional smoothness and $L_p$ -approximation on the Wiener space

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## Abstract

We consider stochastic integral representations of  $g(Y_1)$  with respect to a process  $Y$ , where  $Y$  is the  $d$ -dimensional Brownian motion or the coordinate-wise geometric Brownian motion. For  $2 \leq p < \infty$ , we relate the  $L_p$ -norm of the discretization error of Riemann approximations of this integral to the Besov regularity of  $g(Y_1)$  in the Malliavin sense and to the  $L_p$ -integrability of a Riemann-Liouville type operator.

# 1 Introduction

During the last years approximation problems for stochastic integrals were studied by several authors [26, 16, 10, 7, 11, 8, 18, 19, 13, 14, 22]. One motivation comes from Stochastic Finance, where the problem can be interpreted as a discrete time hedging problem. But the problem is also relevant for the simulation of paths of a stochastic integral and, in general, is of interest as an approximation problem for its own. To explain the purpose of this paper let us introduce some of the notation.

We let  $W = (W_t)_{t \in [0,1]}$  be a standard  $d$ -dimensional Brownian motion starting in zero defined on  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,1]})$ , where  $(\Omega, \mathcal{F}, \mathbb{P})$  is complete and  $(\mathcal{F}_t)_{t \in [0,1]}$  is the augmentation of the natural filtration and where we can assume that  $\mathcal{F} = \mathcal{F}_1$ . As processes driving the stochastic integrals we use the Brownian motion and the coordinate-wise geometric Brownian motion, i.e.

$$Y_t := (W_t^{(1)}, \dots, W_t^{(d)})^\top \quad \text{and} \quad E := \mathbb{R}^d$$

or

$$Y_t := (e^{W_t^{(1)} - \frac{t}{2}}, \dots, e^{W_t^{(d)} - \frac{t}{2}})^\top \quad \text{and} \quad E := (0, \infty)^d.$$

Then we have

$$dY_t = \sigma(Y_t) dW_t,$$

where  $Y$  is considered as a column vector and the  $d \times d$ -matrix  $\sigma(y)$  is given by  $\sigma(y) = I_d$  or  $(\sigma_{ij}(y))_{i,j=1}^d = (\delta_{i,j} y_i)_{i,j=1}^d$ , respectively. The parabolic differential operator associated to the diffusion  $Y$  is

$$\mathcal{A} := \frac{\partial}{\partial t} + \frac{1}{2} \sum_{k=1}^d \sigma_{kk}^2 \frac{\partial^2}{\partial y_k^2}.$$

Given a Borel-function  $g : E \rightarrow \mathbb{R}$  with  $g(Y_1) \in L_2$  we let

$$G(t, y) := \mathbb{E}(g(Y_1) | Y_t = y) \tag{1}$$

and notice that  $G(1, y) = g(y)$ . Integrability properties of  $G$  and its derivatives are given in Lemma A.2 below and are used implicitly in this paper. The function  $G$  solves the backward parabolic PDE

$$\mathcal{A}G = 0 \quad \text{on} \quad [0, 1) \times E.$$

For  $0 \leq s < t < 1$ , Itô's formula implies that

$$G(t, Y_t) - G(s, Y_s) = \int_s^t \nabla G(u, Y_u) \sigma(Y_u) dW_u \text{ a.s.}, \quad (2)$$

where  $\nabla G(t, x)$  is considered as a row vector. Furthermore,

$$g(Y_1) = \mathbb{E}g(Y_1) + \int_0^1 \nabla G(u, Y_u) \sigma(Y_u) dW_u \text{ a.s.} \quad (3)$$

by  $t \uparrow 1$ , where the convergence takes place in  $L_2$  (or later even in  $L_p$  if  $g(Y_1) \in L_p$  with  $2 \leq p < \infty$ ). One purpose of this paper is to investigate Riemann approximations of the stochastic integral in (3) by the following quantities:

**Definition 1.1.** (i) Let  $\mathcal{T}^{\text{rand}}$  be the set of all sequences of stopping times  $\tau = (\tau_i)_{i=0}^n$  with  $0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{n-1} < \tau_n = 1$  where  $n = 1, 2, \dots$ , such that  $\tau_i$  is  $\mathcal{F}_{\tau_{i-1}}$ -measurable for  $i = 1, \dots, n-1$ , i.e.

$$\{\tau_i \in B\} \cap \{\tau_{i-1} \leq t\} \in \mathcal{F}_t \text{ for } t \in [0, 1] \text{ and } B \in \mathcal{B}(\mathbb{R}).$$

(ii) Given a time-net  $\tau = (\tau_i)_{i=0}^n \in \mathcal{T}^{\text{rand}}$ ,  $0 \leq t \leq 1$  and  $g(Y_1) \in L_2$ , we let

$$\begin{aligned} C_t(g(Y_1), \tau) &:= \int_0^t \nabla G(s, Y_s) dY_s \\ &\quad - \sum_{i=1}^n \nabla G(\tau_{i-1}, Y_{\tau_{i-1}})(Y_{\tau_i \wedge t} - Y_{\tau_{i-1} \wedge t}), \\ C_t(g(Y_1), \tau, v) &:= \int_0^t \nabla G(s, Y_s) dY_s - \sum_{i=1}^n v_{\tau_{i-1}}(Y_{\tau_i \wedge t} - Y_{\tau_{i-1} \wedge t}), \end{aligned}$$

where  $v = (v_{\tau_{i-1}})_{i=1}^n$  is a sequence of random row vectors  $v_{\tau_{i-1}} : \Omega \rightarrow \mathbb{R}^d$ .

The study of the approximation error for discretizations of the above type that are based on stopping times is done in the literature often from a different point of view: either one considers weak or stable limits of appropriate rescaled error processes (for instance in [4]) or one studies mean square results of asymptotic nature (for example [20, 8, 3, 5]), where sometimes an average of the number of used stopping times is involved. The main difference to our paper is that we obtain in Theorem 5.1 accurate results for a *fixed*

*finite* sequence of stopping times. This is the reason that we have to use the additional assumption that  $\tau_i$  is  $\mathcal{F}_{\tau_{i-1}}$ -measurable as without this restriction the techniques known so far do not give Theorem 5.1 (see Remark 5.3).

For the rest of this section we restrict ourselves to the simple approximation  $C_t(g(Y_1), \tau)$ , but the paper also deals with the optimal approximation based on  $C_t(g(Y_1), \tau, v)$ . First the strong  $L_2$ -error  $\|C_t(g(Y_1), \tau)\|_{L_2}$  for deterministic nets was considered (see, for example, [16, 10, 7] and for an overview [12]), where Hilbert space techniques can be used. Parallel to this the local  $L_2$ -error was investigated in [11] which yields to weighted BMO-estimates and therefore better tail estimates. In our case, the BMO and the  $L_2$ -estimates form the two end-points of a scale of spaces that contain the  $L_p$ -spaces with  $2 < p < \infty$ . The purpose of this paper is to investigate this  $L_p$ -error. The techniques for the  $L_p$ -estimates differ from the  $L_2$ -estimates because the problem cannot be translated into a one step approximation problem due to the missing Itô-isometry in  $L_2$  that allows to interchange the integral with respect to the time and the integral with respect to the probability measure. In the present paper, that extends [23] from the second author, the following results are obtained:

- (1) In Theorem 5.1 we extend the description of the  $L_2$ -approximation error for deterministic nets to the  $L_p$ -error  $\|C_t(g(Y_1), \tau)\|_p$  with  $2 \leq p < \infty$  and  $\tau \in \mathcal{T}^{\text{rand}}$ .
- (2) From Theorem 5.1 we deduce in Theorem 5.6 a description of the random variables that can be approximated with equidistant time-nets with a rate  $n^{-\theta/2}$  in  $L_p$  for  $0 < \theta < 1$  in terms of the Besov spaces  $\mathbb{B}_{p,\infty}^\theta$ .
- (3) Looking for the optimal approximation by adapted (deterministic) time-nets we get a characterization using a Riemann-Liouville operator of integration in Theorem 5.4. This operator is introduced in Section 4.
- (4) In order to obtain Theorem 5.6 and the connection between Theorem 5.4 and the Besov spaces  $\mathbb{B}_{p,q}^\theta$ , an appropriate characterization of the interpolation spaces on the Wiener space is provided in Theorem 3.1.

## 2 Preliminaries

We use  $A \sim_c B$  for  $A/c \leq B \leq cA$  whenever  $A, B \geq 0$  and  $c \geq 1$ , and let  $|\cdot|$  be the Euclidean norm for a vector or the Hilbert-Schmidt norm for a matrix. Given a random vector or a random matrix  $A$ , we write  $\|A\|_{L_p} := \| \|A\| \|_{L_p}$  and denote the transpose of  $A$  by  $A^\top$ . Let us recall the real interpolation method that we use to generate the Malliavin Besov spaces.

**Definition 2.1** ([1], [2]). *Let  $(X_0, X_1)$  be a compatible couple of Banach spaces. Given  $x \in X_0 + X_1$  and  $\lambda > 0$ , the  $K$ -functional is defined by*

$$K(x, \lambda; X_0, X_1) := \inf \{ \|x_0\|_{X_0} + \lambda \|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i \}.$$

*Given  $0 < \theta < 1$  and  $1 \leq q \leq \infty$ , we let  $(X_0, X_1)_{\theta, q}$  be the space of all  $x \in X_0 + X_1$  such that*

$$\|x\|_{(X_0, X_1)_{\theta, q}} := \left\| \lambda^{-\theta} K(x, \lambda; X_0, X_1) \right\|_{L_q((0, \infty), \frac{d\lambda}{\lambda})} < \infty.$$

Our Wiener space is constructed in a standard way (see, for example, [21]): For  $H = \ell_2^d$  and  $(M, \Sigma, \mathbb{Q}) = (\mathbb{R}^d, \overline{\mathcal{B}(\mathbb{R}^d)}, \gamma_d)$ , where  $\gamma_d$  is the  $d$ -dimensional standard Gaussian measure, we let  $g_a : M \rightarrow \mathbb{R}$  be given by

$$g_a(x) := \langle x, a \rangle_{\ell_2^d}$$

and obtain an iso-normal family  $(g_a)_{a \in H}$  of centered Gaussian random variables. Let  $\mathbb{D}_{1,2} \subseteq L_2$  be the Malliavin Sobolev space and

$$D : \mathbb{D}_{1,2} \rightarrow L_2^H(M, \Sigma, \mu)$$

the Malliavin derivative. Given  $2 \leq p < \infty$ , the Banach space  $\mathbb{D}_{1,p} \subseteq L_p$  is given by

$$\mathbb{D}_{1,p} := \left\{ f \in \mathbb{D}_{1,2} : \|f\|_{\mathbb{D}_{1,p}} := \left( \|f\|_{L_p}^p + \|Df\|_{L_p^H}^p \right)^{\frac{1}{p}} < \infty \right\}.$$

Here and later we use the convention  $\|f\|_{L_p} = \|f\|_{L_p(\mathbb{R}^d, \gamma_d)}$  and  $\|Df\|_{L_p^H} = \|Df\|_{L_p^H(\mathbb{R}^d, \gamma_d)}$

**Definition 2.2.** *For  $0 < \theta < 1$  and  $1 \leq q \leq \infty$  we let*

$$\mathbb{B}_{p,q}^\theta := (L_p, \mathbb{D}_{1,p})_{\theta, q}$$

*be the Malliavin Besov space of fractional smoothness  $\theta$  and fine-index  $q$ .*

We use the Burkholder-Davis-Gundy inequality for Brownian martingales with values in a separable Hilbert space. An explicit formulation is as follows: Assume for  $i = 1, 2, \dots$  progressively measurable processes  $(L_t^i)_{t \in [0,1]}$  with  $L_t^i : \Omega \rightarrow \mathbb{R}^d$  considered as row vectors and such that

$$\sum_{i=1}^{\infty} \mathbb{E} \int_0^1 |L_t^i|^2 dt < \infty,$$

then, for all  $1 < p < \infty$ , there is a constant  $c_{(4)} = c_{(4)}(p) \geq 1$  such that

$$\left\| \left( \sum_{i=1}^{\infty} \left| \int_0^1 L_u^i dW_u \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \sim_{c_{(4)}} \left\| \left( \sum_{i=1}^{\infty} \int_0^1 |L_u^i|^2 du \right)^{\frac{1}{2}} \right\|_{L_p}. \quad (4)$$

### 3 Fractional smoothness on the Wiener space

In this section we characterize the Malliavin Besov spaces  $\mathbb{B}_{p,q}^{\theta}$  by the behavior of  $G$  from (1) in the case  $Y = W$ . To make this more clear we do a change of notation and replace  $g$  by  $f$  and  $G$  by  $F$ . This means that  $f \in L_2(\mathbb{R}^d, \gamma_d)$  and

$$F(t, x) := \mathbb{E} f(x + W_{1-t}) \quad \text{for } (t, x) \in [0, 1] \times \mathbb{R}^d.$$

We also use the Hessian  $d \times d$  matrix

$$D^2 F := \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{i,j=1}^d.$$

It is known that

$$f \in \mathbb{D}_{1,2} \quad \text{if and only if} \quad \int_0^1 \|D^2 F(t, W_t)\|_{L_2}^2 dt < \infty. \quad (5)$$

Moreover, for all  $f \in L_2(\mathbb{R}^d, \gamma_d)$  we have

$$\nabla F(t, W_t) = \nabla F(0, 0) + \left( \int_0^t D^2 F(u, W_u) dW_u \right)^{\top} \quad \text{a.s.} \quad (6)$$

for  $0 \leq t < 1$ , where  $\nabla F(t, \cdot)$  is considered as a row vector. If  $f \in \mathbb{D}_{1,2}$ , then (6) can be extended to  $t = 1$  with the convention  $\nabla F(1, \cdot) := Df$ . Now we generalize this relation to the scale of Besov spaces.

**Theorem 3.1.** *Let  $2 \leq p < \infty$ ,  $0 < \theta < 1$ ,  $1 \leq q \leq \infty$  and  $f \in L_p(\mathbb{R}^d, \gamma_d)$ . Then*

$$\begin{aligned} \|f\|_{\mathbb{B}_{p,q}^\theta} &\sim_{c(3.1)} \|f\|_{L_p} + \left\| (1-t)^{-\frac{\theta}{2}} \|F(1, W_1) - F(t, W_t)\|_{L_p} \right\|_{L_q([0,1], \frac{dt}{1-t})} \\ &\sim_{c(3.1)} \|f\|_{L_p} + \left\| (1-t)^{\frac{1-\theta}{2}} \|\nabla F(t, W_t)\|_{L_p} \right\|_{L_q([0,1], \frac{dt}{1-t})} \\ &\sim_{c(3.1)} \|f\|_{L_p} + \left\| (1-t)^{\frac{2-\theta}{2}} \|D^2 F(t, W_t)\|_{L_p} \right\|_{L_q([0,1], \frac{dt}{1-t})}, \end{aligned}$$

where  $c(3.1) \geq 1$  depends at most on  $(p, \theta, q)$ .

**Remark 3.2.** Theorem 3.1 generalizes [13, Theorem 2.2], where  $p = 2$  was considered, and [23, Lemma 4.7], which was proved for  $2 < p < \infty$  and  $q = \infty$ .

To prove Theorem 3.1 we start with

**Proposition 3.3.** *Let  $2 \leq p < \infty$ . There exists a constant  $c(3.3) \geq 1$  depending at most on  $p$  such that for any  $0 < t < 1$ ,*

$$K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p}) \sim_{c(3.3)} \left( \|f(W_1) - F(t, W_t)\|_{L_p} + \sqrt{1-t} \|f\|_{L_p} \right).$$

*Proof.* (a) Fix  $0 < t < 1$  and  $\varepsilon > 0$ . We find  $f_0 \in L_p$  and  $f_1 \in \mathbb{D}_{1,p}$  such that  $f = f_0 + f_1$  and

$$\|f_0\|_{L_p} + \sqrt{1-t} \|f_1\|_{\mathbb{D}_{1,p}} \leq K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p}) + \varepsilon.$$

For  $F_i(t, x) := \mathbb{E}(f_i(W_1)|W_t = x)$  we obtain from (2) that

$$\begin{aligned} &\|f(W_1) - F(t, W_t)\|_{L_p} \\ &\leq \|f_0(W_1) - F_0(t, W_t)\|_{L_p} + \|f_1(W_1) - F_1(t, W_t)\|_{L_p} \\ &= \|f_0(W_1) - F_0(t, W_t)\|_{L_p} + \left\| \int_t^1 \nabla F_1(u, W_u) dW_u \right\|_{L_p} \\ &\leq \|f_0(W_1) - F_0(t, W_t)\|_{L_p} + c_{(4)} \left( \int_t^1 \|\nabla F_1(u, W_u)\|_{L_p}^2 du \right)^{\frac{1}{2}} \\ &\leq 2\|f_0\|_{L_p} + c_{(4)} \sqrt{1-t} \|f_1\|_{\mathbb{D}_{1,p}} \\ &\leq c[K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p}) + \varepsilon], \end{aligned}$$

where  $c_{(4)} \geq 1$  is the constant from the Burkholder-Davis-Gundy inequalities,  $c := \max\{c_{(4)}, 2\}$  and we employed the facts that  $2 \leq p < \infty$  and that (6) gives

$$\|\nabla F_1(u, W_u)\|_{L_p} \leq \|f_1\|_{\mathbb{D}_{1,p}} \quad \text{for all } 0 \leq u \leq 1.$$

Letting  $\epsilon \rightarrow 0$  and observing that  $\sqrt{1-t}\|f\|_{L_p} \leq K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p})$  we achieve the first part of the desired inequality.

(b) For  $0 < t < 1$  we set

$$g_t(x) := F(t, \sqrt{t}x) \quad \text{and} \quad h_t(x) := f(x) - F(t, \sqrt{t}x)$$

so that

$$\begin{aligned} \|g_t\|_{\mathbb{D}_{1,p}}^p &= \left\| F(t, \sqrt{t}x) \right\|_{L_p}^p + \left\| \nabla F(t, \sqrt{t}x) \sqrt{t} \right\|_{L_p}^p \\ &\leq \|f\|_{L_p}^p + \|\nabla F(t, W_t)\|_{L_p}^p. \end{aligned}$$

Applying (2) for  $Y = W$ , the Burkholder-Davis-Gundy inequalities, the fact that  $\|\nabla F(t, W_t)\|_{L_p}$  is non-decreasing in  $t$  and that  $2 \leq p < \infty$ , we estimate

$$\begin{aligned} \|F(t, W_t)\|_{L_p} &\leq \left\| \int_0^t \nabla F(u, W_u) dW_u \right\|_{L_p} + \|F(0, W_0)\|_{L_p} \\ &\leq c_{(4)} \|\nabla F(t, W_t)\|_{L_p} + |\mathbb{E}f(W_1)|. \end{aligned}$$

Thus

$$\begin{aligned} &\|g_t\|_{\mathbb{D}_{1,p}} \\ &\leq \|f(W_1) - F(t, W_t)\|_{L_p} + \|F(t, W_t)\|_{L_p} + \|\nabla F(t, W_t)\|_{L_p} \\ &\leq \|f(W_1) - F(t, W_t)\|_{L_p} + (1 + c_{(4)}) \|\nabla F(t, W_t)\|_{L_p} + |\mathbb{E}f(W_1)| \\ &\leq \left[ 1 + (1 + c_{(4)})c_{(A.3)}(1-t)^{-\frac{1}{2}} \right] \|f(W_1) - F(t, W_t)\|_{L_p} + |\mathbb{E}f(W_1)|, \end{aligned}$$

where we used Lemma A.3. Exploiting an independent Brownian motion  $\tilde{W}$  and that the covariance structures of  $(W_1, \sqrt{t}W_1 + \sqrt{1-t}\tilde{W}_1)$  and  $(W_1, W_{\sqrt{t}} + \tilde{W}_{1-\sqrt{t}})$  are the same, we obtain for  $h_t$  that

$$\begin{aligned} \|h_t\|_{L_p} &= \left[ \mathbb{E} \left| f(W_1) - \tilde{\mathbb{E}}f(\sqrt{t}W_1 + \sqrt{1-t}\tilde{W}_1) \right|^p \right]^{\frac{1}{p}} \\ &\leq \left[ \mathbb{E} \tilde{\mathbb{E}} \left| f(W_1) - f(W_{\sqrt{t}} + \tilde{W}_{1-\sqrt{t}}) \right|^p \right]^{\frac{1}{p}} \end{aligned}$$



$$\begin{aligned}
&\leq \left\| f(W_1) - F(\sqrt{t}, W_{\sqrt{t}}) \right\|_{L_p} \\
&\quad + \left\| F(\sqrt{t}, W_{\sqrt{t}}) - f(W_{\sqrt{t}} + \tilde{W}_{1-\sqrt{t}}) \right\|_{L_p} \\
&= 2 \left\| f(W_1) - F(\sqrt{t}, W_{\sqrt{t}}) \right\|_{L_p} \\
&\leq 2 \left\| f(W_1) - F(t, W_t) \right\|_{L_p} + 2 \left\| F(t, W_t) - F(\sqrt{t}, W_{\sqrt{t}}) \right\|_{L_p} \\
&\leq 4 \left\| f(W_1) - F(t, W_t) \right\|_{L_p}.
\end{aligned}$$

Hence

$$\begin{aligned}
&K(f, \sqrt{1-t}; L_p, \mathbb{D}_{1,p}) \\
&\leq \|h_t\|_{L_p} + (1-t)^{\frac{1}{2}} \|g_t\|_{\mathbb{D}_{1,p}} \\
&\leq 4 \left\| f(W_1) - F(t, W_t) \right\|_{L_p} + (1-t)^{\frac{1}{2}} \left[ |\mathbb{E}f(W_1)| \right. \\
&\quad \left. + [1 + (1+c_{(4)})c_{(A.3)}(1-t)^{-\frac{1}{2}}] \left\| f(W_1) - F(t, W_t) \right\|_{L_p} \right] \\
&\leq (1-t)^{\frac{1}{2}} |\mathbb{E}f(W_1)| + \\
&\quad + [5 + (1+c_{(4)})c_{(A.3)}] \left\| f(W_1) - F(t, W_t) \right\|_{L_p}
\end{aligned}$$

and the proof is complete.  $\square$

We are now ready to prove the main result of this section.

*Proof of Theorem 3.1.* To verify the assumptions of Proposition A.4, we set

$$\begin{aligned}
d^0(t) &:= \|f(W_1) - F(t, W_t)\|_{L_p}, \\
d^1(t) &:= \|\nabla F(t, W_t)\|_{L_p}, \\
d^2(t) &:= \|D^2 F(t, W_t)\|_{L_p},
\end{aligned}$$

$A := 2c_{(A.3)} \|f\|_{L_p}$  and  $\alpha := c_{(4)} \vee c_{(A.3)}$ , where  $c_{(4)} \geq 1$  is taken from (4). Then Lemma A.3 implies that

$$d^k(t) \leq c_{(A.3)} (1-t)^{-\frac{k}{2}} d^0(t) \quad \text{for } k = 1, 2.$$

By (2), (6), the Burkholder-Davis-Gundy inequalities (4) and  $2 \leq p < \infty$ , we also see that

$$d^0(t) \leq c_{(4)} \left( \int_0^t \|\nabla F(s, W_s)\|_{L_p}^2 ds \right)^{\frac{1}{2}} = c_{(4)} \left( \int_0^t [d^1(s)]^2 ds \right)^{\frac{1}{2}}$$

and

$$\begin{aligned}
d^1(t) &\leq \|\nabla F(0, W_0)\|_{L_p} + \left\| \int_0^t D^2 F(s, W_s) dW_s \right\|_{L_p} \\
&\leq 2c_{(A.3)} \|f\|_{L_p} + c_{(4)} \left( \int_0^t \|D^2 F(s, W_s)\|_{L_p}^2 ds \right)^{\frac{1}{2}} \\
&= 2c_{(A.3)} \|f\|_{L_p} + c_{(4)} \left( \int_0^t [d^2(s)]^2 ds \right)^{\frac{1}{2}},
\end{aligned}$$

where we used Lemma A.3. Now, applying (20) gives the equivalence between the last three expressions in Theorem 3.1. It remains to check that

$$\|f\|_{\mathbb{B}_{p,q}^\theta} \sim_c \|f\|_{L_p} + \left\| (1-t)^{-\frac{\theta}{2}} \|F(1, W_1) - F(t, W_t)\|_{L_p} \right\|_{L_q([0,1], \frac{dt}{1-t})}$$

for some  $c = c(p, q, \theta) \geq 1$ , which follows from Proposition 3.3.  $\square$

**Remark 3.4.** In the literature the interpolation spaces on the Wiener space are considered, for example, in [25, 17, 13, 14, 23]. A classical approach is based on semi-groups (here the Ornstein-Uhlenbeck semi-group) which yields to a scaling that is different from our scaling (see [2, Theorem 6.7.3 and p. 167, Exercise 28] and [24, Sections 1.13.2, 1.14.5 and 1.15.2]).

## 4 The Riemann-Liouville operator $D^{Y,\theta}$

Riemann-Liouville type operators are typically used to describe fractional regularity. We use these operators to replace the Besov regularity defined by real interpolation when we consider the approximation by adapted time-nets in Theorem 5.4 below. The operator, introduced in the following Definition 4.1, was also used in a slightly modified form in [14], where the weak convergence of the error processes was considered.

**Definition 4.1.** For  $g(Y_1) \in L_2$ ,  $0 < \theta \leq 1$  and  $0 \leq t \leq 1$ , we let

$$D_t^{Y,\theta} g(Y_1) := \left( \int_0^t (1-u)^{1-\theta} H_G^2(u, Y_u) du \right)^{\frac{1}{2}},$$

where

$$H_G^2(u, y) := \sum_{k,l=1}^d \left| \left( \sigma_{kk} \sigma_{ll} \frac{\partial^2 G}{\partial y_k \partial y_l} \right) (u, y) \right|^2.$$

From now on we use the following convention: For  $0 \leq t \leq 1$  we let

$$y_k(t) = \begin{cases} x_k & : Y = W \\ e^{x_k - \frac{t}{2}} & : \text{else} \end{cases} \quad \text{and} \quad y(t) := (y_1(t), \dots, y_d(t))^\top, \quad (7)$$

and define the functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $F : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$  as

$$f(x) := g(y(1)) \quad \text{and} \quad F(t, x) := \mathbb{E}f(x + W_{1-t}) \quad (8)$$

so that  $f(W_1) = g(Y_1)$  and  $F(t, x) = G(t, y(t))$ . In the case that  $Y$  is the coordinate-wise geometric Brownian motion, this notation implies that

$$y_k(t)y_l(t)\frac{\partial^2 G}{\partial y_k \partial y_l}(t, y(t)) = \frac{\partial^2 F}{\partial x_k \partial x_l}(t, x) - \delta_{kl}\frac{\partial F}{\partial x_k}(t, x) \quad (9)$$

for  $k, l = 1, \dots, d$ . Let us summarize the connections between the Besov spaces and the operator  $D^{Y, \theta}$  known to us:

**Proposition 4.2.** *For  $g(Y_1) \in L_p$  with  $2 \leq p < \infty$  the following assertions hold true:*

- (i) *If  $2 < p < \infty$  and  $0 < \theta < 1$ , then*
  - (a)  *$f \in \mathbb{B}_{p,2}^\theta$  implies  $D_1^{Y, \theta}g(Y_1) \in L_p$ ,*
  - (b)  *$D_1^{Y, \theta}g(Y_1) \in L_p$  implies  $f \in \mathbb{B}_{p,\infty}^\theta$ .*
- (ii) *If  $2 \leq p < \infty$ , then  $D_1^{Y,1}g(Y_1) \in L_p$  if and only if  $f \in \mathbb{D}_{1,p}$ .*
- (iii) *If  $0 < \theta < 1$ , then  $D_1^{Y, \theta}g(Y_1) \in L_2$  if and only if  $f \in \mathbb{B}_{2,2}^\theta$ .*

*Proof.* (i, part a) Because of  $2 \leq p < \infty$ , we see that

$$\begin{aligned} \left\| D_1^{Y, \theta}g(Y_1) \right\|_{L_p} &= \left\| \left( \int_0^1 (1-t)^{1-\theta} H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\leq \left( \int_0^1 (1-t)^{1-\theta} \|H_G(t, Y_t)\|_{L_p}^2 dt \right)^{\frac{1}{2}} \\ &= \left\| (1-t)^{\frac{2-\theta}{2}} \|H_G(t, Y_t)\|_{L_p} \right\|_{L_2([0,1], \frac{dt}{1-t})}. \end{aligned}$$

Theorem 3.1 completes the proof, since in the case that  $Y$  is the Brownian motion, we have  $H_G(t, Y_t) = |D^2 F(t, W_t)|$  and in the other case, we can use (9) and Theorem 3.1 again to see that

$$\left\| (1-t)^{\frac{2-\theta}{2}} \|H_G(t, Y_t)\|_{L_p} \right\|_{L_2([0,1], \frac{dt}{1-t})} < \infty.$$

(i, part b) For all  $0 < t \leq 1$ ,

$$\begin{aligned} \left\| D_1^{Y,\theta} g(Y_1) \right\|_{L_p} &\geq \left\| \left( \int_0^t (1-s)^{1-\theta} H_G^2(s, Y_s) ds \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\geq (1-t)^{\frac{1-\theta}{2}} \left\| \left( \int_0^t H_G^2(s, Y_s) ds \right)^{\frac{1}{2}} \right\|_{L_p}. \end{aligned}$$

If  $Y$  is the Brownian motion, then we can bound this from below by

$$\frac{1}{c_{(4)}} (1-t)^{\frac{1-\theta}{2}} \|\nabla F(t, W_t) - \nabla F(0, W_0)\|_{L_p},$$

where we have used (4) and (6). This implies that

$$\|\nabla F(t, W_t)\|_{L_p} \leq \|\nabla F(0, W_0)\|_{L_p} + c_{(4)} (1-t)^{\frac{\theta-1}{2}} \left\| D_1^{W,\theta} f(W_1) \right\|_{L_p}$$

and Theorem 3.1 can be used again. If  $Y$  is the coordinate-wise geometric Brownian motion, then we get from (9) that

$$\begin{aligned} &\left\| \left( \int_0^t H_G^2(s, Y_s) ds \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\geq \left\| \left( \int_0^t H_F^2(s, W_s) ds \right)^{\frac{1}{2}} \right\|_{L_p} - \left\| \left( \int_0^1 |\nabla F(s, W_s)|^2 ds \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\geq \frac{1}{c_{(4)}} \|\nabla F(t, W_t) - \nabla F(0, W_0)\|_{L_p} - \left\| \left( \int_0^1 |\nabla F(s, W_s)|^2 ds \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\geq \frac{1}{c_{(4)}} \|\nabla F(t, W_t)\|_{L_p} - \frac{1}{c_{(4)}} \|\nabla F(0, W_0)\|_{L_p} \end{aligned}$$

$$- \left\| \left( \int_0^1 |\nabla F(s, W_s)|^2 ds \right)^{\frac{1}{2}} \right\|_{L_p}$$

where we again used the Burkholder-Davis-Gundy inequalities. Because the last two terms on the right-hand side are finite, we can conclude as in the case of the Brownian motion.

(ii) Because of (9) and  $\left( \int_0^1 |\nabla F(t, W_t)|^2 dt \right)^{\frac{1}{2}} \in L_p$  we get  $D_1^{Y,1} g(Y_1) \in L_p$  if and only if  $\left( \int_0^1 |D^2 F(t, W_t)|^2 dt \right)^{\frac{1}{2}} \in L_p$ . Using relations (5) and (6) one easily checks that this is equivalent to  $f \in \mathbb{D}_{1,p}$ .

(iii) Since (9) implies that

$$\left\| D_1^{Y,\theta} g(Y_1) \right\|_{L_2}^2 = \int_0^1 (1-t)^{1-\theta} \|H_G(t, Y_t)\|_{L_2}^2 dt < \infty$$

if and only if  $\int_0^1 (1-t)^{1-\theta} \|D^2 F(t, W_t)\|_{L_2}^2 dt < \infty$ , we can use Theorem 3.1.  $\square$

## 5 An approximation problem in $L_p$

In the whole section we use the conventions (7) and (8).

**Time-nets.** Given a deterministic sequence  $0 = t_0 \leq \dots \leq t_{n-1} < t_n = 1$  and  $0 < \theta \leq 1$ , we let

$$\begin{aligned} |(t_i)_{i=0}^n|_\theta &:= \sup_{i=1,\dots,n} \sup_{t_{i-1} \leq u < t_i} \frac{|t_i - u|}{(1-u)^{1-\theta}} = \sup_{i=1,\dots,n} \frac{|t_i - t_{i-1}|}{(1-t_{i-1})^{1-\theta}}, \\ |(t_i)_{i=0}^n| &:= |(t_i)_{i=0}^n|_1 \end{aligned}$$

so that  $|(t_i)_{i=0}^n|$  is the usual mesh-size. As special adapted deterministic time-nets we use  $\tau_n^\theta = (t_{i,n}^\theta)_{i=0}^n$  defined by

$$t_{i,n}^\theta := 1 - \left( 1 - \frac{i}{n} \right)^{\frac{1}{\theta}}.$$

For these time-nets,

$$|t_{i,n}^\theta - u| \leq \frac{|t_{i,n}^\theta - u|}{(1-u)^{1-\theta}} \leq \frac{|t_{i,n}^\theta - t_{i-1,n}^\theta|}{(1-t_{i-1,n}^\theta)^{1-\theta}} \leq \frac{1}{\theta n} \quad \text{for } u \in [t_{i-1,n}^\theta, t_{i,n}^\theta] \quad (10)$$

which implies that

$$|\tau_n^\theta| \leq |\tau_n^\theta|_\theta \leq \frac{1}{\theta n}. \quad (11)$$

Moreover, we have that

$$\frac{(1 - t_{i-1,n}^\theta)^{1-\theta}}{|t_{i,n}^\theta - t_{i-1,n}^\theta|} \leq \beta n \quad (12)$$

for some  $\beta > 0$  independent from  $n$ .

**The basic equivalence in  $L_p$ .** The following result reduces the computation of the  $L_p$ -norm of the error processes defined in Definition 1.1 to an expression involving  $H_G(t, Y_t)$  similar to a square function. This result generalizes [10, Theorem 4.4] proved for deterministic nets in the  $L_2$ -case.

**Theorem 5.1.** *For  $2 \leq p < \infty$  there is a constant  $c_{(5.1)} \geq 1$  depending at most on  $p$  such that for all  $g(Y_1) \in L_p$  and  $\tau = (\tau_i)_{i=0}^n \in \mathcal{T}^{\text{rand}}$  one has that*

$$\begin{aligned} \|C_1(g(Y_1), \tau)\|_{L_p} &\leq c_{(5.1)} \left\| \left( \sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p}, \\ \inf_v \|C_1(g(Y_1), \tau, v)\|_{L_p} &\geq \frac{1}{c_{(5.1)}} \left\| \left( \sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p}, \end{aligned}$$

where the infimum is taken over all simple random vectors  $v_{\tau_{i-1}} : \Omega \rightarrow \mathbb{R}^d$  that are  $\mathcal{F}_{\tau_{i-1}}$ -measurable.

**Remark 5.2.** Both inequalities in Theorem 5.1 are proved by stopping at  $0 < T < 1$  and letting  $T \uparrow 1$ . Therefore, it might be possible for one or both sides of an inequality to be infinite. However, this cannot be the case: Step (b) of our proof for the trivial time-net  $0 = t_0 < t_1 = 1$  gives by (14) that

$$\begin{aligned} &\left\| \left( \int_0^T (T - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\leq c \|\mathbb{E}(g(Y_1) | \mathcal{F}_T) - \mathbb{E}g(Y_1) - \nabla G(0, Y_0)(Y_T - Y_0)\|_{L_p} \\ &\leq c \|g(Y_1) - \mathbb{E}g(Y_1) - \nabla G(0, Y_0)(Y_1 - Y_0)\|_{L_p} < \infty \end{aligned}$$

so that

$$\left\| \left( \sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \leq \left\| \left( \int_0^1 (1-t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} < \infty.$$

Following (15) from step (c), this implies that  $\sup_{0 \leq T < 1} \|C_T(g(Y_1), \tau)\|_{L_p} < \infty$ , from which we can conclude that

$$\left( \int_0^1 \sum_{i=1}^n \chi_{(\tau_{i-1}, \tau_i]}(t) |\nabla G(\tau_{i-1}, Y_{\tau_{i-1}}) \sigma(Y_t)|^2 dt \right)^{\frac{1}{2}} \in L_p$$

and  $C_1(g(Y_1), \tau) \in L_p$ . Finally, we have that

$$\inf_v \|C_1(g(Y_1), \tau, v)\|_{L_p} \leq \|C_1(g(Y_1), \tau)\|_{L_p}, \quad (13)$$

where the infimum is taken over all simple random vectors  $v_{\tau_{i-1}} : \Omega \rightarrow \mathbb{R}^d$  that are  $\mathcal{F}_{\tau_{i-1}}$ -measurable. The latter also implies that all three expressions - in particular the simple and optimal  $L_p$ -approximation - in Theorem 5.1 are equivalent up to a multiplicative constant.

*Proof of Theorem 5.1.* (a) Assume a deterministic time  $0 < T < 1$ , two stopping times  $0 \leq a \leq b \leq T$  and that  $v_a$  is a simple  $\mathcal{F}_a$ -measurable random (row) vector. Exploiting relations (6) and (9) one quickly checks that

$$\left( \frac{\partial G}{\partial y_k}(b, Y_b) - v_a^k \right) \sigma_{kk}(Y_b) = m_a(k) + \sum_{l=1}^d \int_a^b \lambda_u^a(k, l) dW_u^l$$

with

$$\begin{aligned} m_a(k) &:= \left( \frac{\partial G}{\partial y_k}(a, Y_a) - v_a^k \right) \sigma_{kk}(Y_a) \text{ and} \\ \lambda_u^a(k, l) &:= \left( \sigma_{kk} \sigma_{ll} \frac{\partial^2 G}{\partial y_k \partial y_l} \right) (u, Y_u) + \left( \frac{\partial G}{\partial y_l}(u, Y_u) - v_a^l \right) \left( \sigma_{ll} \frac{\partial \sigma_{ll}}{\partial y_k} \right) (Y_u), \end{aligned}$$

where  $m_a := (m_a(1), \dots, m_a(d))$  will be considered as a row vector.

(b) Lower bound for  $\|C_1(g(Y_1), \tau, v)\|_{L_p}$ : Let us fix  $0 < T < 1$  and define  $\rho_i := \tau_i \wedge T$  and  $\alpha_{\rho_{i-1}} := v_{\tau_{i-1}} \chi_{\{\tau_{i-1} < T\}}$ . Note that  $\rho_i$  and  $\alpha_{\rho_{i-1}}$  are  $\mathcal{F}_{\rho_{i-1}}$ -measurable for  $i = 1, \dots, n$ . Replacing  $v$  by  $\alpha$  in the definitions of  $m$  and  $\lambda$

from step (a), it follows that

$$C_T(g(Y_1), \tau, v) = \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} \left[ m_{\rho_{i-1}} + \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top \right] dW_t.$$

Using (4) and the convexity inequality [6, pp. 104-105, p. 171], we achieve

$$\begin{aligned} & \left\| \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} \left[ m_{\rho_{i-1}} + \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top \right] dW_t \right\|_{L_p}^p \\ & \geq c_{(4)}^{-p} \mathbb{E} \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} \left| m_{\rho_{i-1}} + \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top \right|^2 dt \right)^{\frac{p}{2}} \\ & \geq c_{(4)}^{-p} (p/2)^{-p/2} \mathbb{E} \left( \sum_{i=1}^n \mathbb{E}_{\mathcal{F}_{\rho_{i-1}}} \int_{\rho_{i-1}}^{\rho_i} \left| m_{\rho_{i-1}} + \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top \right|^2 dt \right)^{\frac{p}{2}} \\ & \geq (c_{(4)} \sqrt{p/2})^{-p} \mathbb{E} \left( \sum_{i=1}^n \mathbb{E}_{\mathcal{F}_{\rho_{i-1}}} \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2 dt \right)^{\frac{p}{2}} \\ & = (c_{(4)} \sqrt{p/2})^{-p} \mathbb{E} \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2 dt \right)^{\frac{p}{2}}, \end{aligned}$$

where we used that  $\rho_i$  is  $\mathcal{F}_{\rho_{i-1}}$ -measurable. From this we deduce that

$$\begin{aligned} & \left\| \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top dW_t \right\|_{L_p} \\ & \leq \left\| \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} m_{\rho_{i-1}} dW_t \right\|_{L_p} \\ & \quad + \left\| \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} \left[ m_{\rho_{i-1}} + \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top \right] dW_t \right\|_{L_p} \\ & \leq c_{(4)} \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p} \\ & \quad + \left\| \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} \left[ m_{\rho_{i-1}} + \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top \right] dW_t \right\|_{L_p} \end{aligned}$$



$$\leq [c_{(4)}^2 \sqrt{p/2} + 1] \left\| \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} \left[ m_{\rho_{i-1}} + \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top \right] dW_t \right\|_{L_p}$$

so that

$$\|C_T(g(Y_1), \tau, v)\|_{L_p} \geq [c_{(4)}^2 \sqrt{p/2} + 1]^{-1} \left\| \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top dW_t \right\|_{L_p}.$$

We continue by writing

$$\begin{aligned} & \left\| \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top dW_t \right\|_{L_p} \\ &= \left\| \sum_{i=1}^n \int_0^1 \chi_{(\rho_{i-1}, \rho_i]}(t) \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} \lambda_u^{\rho_{i-1}} dW_u \right]^\top dW_t \right\|_{L_p} \\ &= \left\| \sum_{i=1}^n \int_0^1 \left[ \int_{\rho_{i-1}}^{t \vee \rho_{i-1}} [\chi_{(\rho_{i-1}, \rho_i]}(t) \lambda_u^{\rho_{i-1}}] dW_u \right]^\top dW_t \right\|_{L_p} \\ &= \left\| \sum_{i=1}^n \int_0^1 \left[ \int_0^t [\chi_{(\rho_{i-1}, t \vee \rho_{i-1}]}(u) \chi_{(\rho_{i-1}, \rho_i]}(t) \lambda_u^{\rho_{i-1}}] dW_u \right]^\top dW_t \right\|_{L_p} \\ &= \left\| \int_0^1 \left[ \int_0^t \mu^\rho(t, u) dW_u \right]^\top dW_t \right\|_{L_p} \end{aligned}$$

with the  $d \times d$ -matrix

$$\mu^\rho(t, u) := \sum_{i=1}^n \chi_{(\rho_{i-1}, t]}(u) \chi_{(\rho_{i-1}, \rho_i]}(t) \lambda_u^{\rho_{i-1}} = \sum_{i=1}^n \chi_{\{\rho_{i-1} < u \leq t \leq \rho_i\}} \lambda_u^{\rho_{i-1}}.$$

Here we used again the condition that  $\rho_i$  is  $\mathcal{F}_{\rho_{i-1}}$ -measurable. By (4) and Lemma A.1,

$$\left\| \int_0^1 \left[ \int_0^t \mu^\rho(t, u) dW_u \right]^\top dW_t \right\|_{L_p}$$

$$\begin{aligned}
& \sim_{c_{(4)}} \left\| \left( \int_0^1 \left| \int_0^1 \mu^\rho(t, u) dW_u \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p} \\
& \sim_{c_{(A.1)}} \left\| \left( \int_0^1 \int_0^t |\mu^\rho(t, u)|^2 du dt \right)^{\frac{1}{2}} \right\|_{L_p} \\
& = \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) |\lambda_t^{\rho_{i-1}}|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p}.
\end{aligned}$$

Letting  $\delta = 0$  if  $Y = W$  and  $\delta = 1$  if  $Y$  is the geometric Brownian motion, this can be combined with

$$\begin{aligned}
& \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \\
& - \delta \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) |(\nabla G(t, Y_t) - \alpha_{\rho_{i-1}}) \sigma(Y_t)|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p} \\
& \leq \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) |\lambda_t^{\rho_{i-1}}|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p} \\
& \leq \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \\
& + \delta \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) |(\nabla G(t, Y_t) - \alpha_{\rho_{i-1}}) \sigma(Y_t)|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p}
\end{aligned}$$

so that

$$\begin{aligned}
& \|C_T(g(Y_1), \tau, v)\|_{L_p} \\
& \geq [c_{(4)}^2 \sqrt{p/2} + 1]^{-1} c_{(4)}^{-1} c_{(A.1)}^{-1} \left[ \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \right.
\end{aligned}$$

$$- \delta \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} |(\nabla G(t, Y_t) - \alpha_{\rho_{i-1}}) \sigma(Y_t)|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p} \right].$$

In the case of the Brownian motion the last term disappears. In the case of the geometric Brownian motion we apply again the Burkholder-Davies-Gundy inequalities to see that

$$\left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} |(\nabla G(t, Y_t) - \alpha_{\rho_{i-1}}) \sigma(Y_t)|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p} \leq c_{(4)} \|C_T(g(Y_1), \tau, v)\|_{L_p}.$$

Hence, in both cases, we have that

$$\begin{aligned} & \inf_v \|C_T(g(Y_1), \tau, v)\|_{L_p} \\ & \geq \frac{1}{[c_{(4)}^2 \sqrt{p/2} + 1] c_{(4)} c(A.1) + c_{(4)}} \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p}. \end{aligned} \quad (14)$$

By  $T \uparrow 1$  we obtain the lower bound of our theorem.

(c) Upper bound for  $\|C_1(g(Y_1), \tau)\|_{L_p}$ : For  $0 < T < 1$ , using the arguments and notation from step (b) and

$$\nu_u^{\rho_{i-1}} := \left( \nabla G(u, Y_u) - \nabla G(\tau_{i-1}, Y_{\tau_{i-1}}) \chi_{\{\tau_{i-1} < T\}} \right) \sigma(Y_u),$$

we obtain

$$\begin{aligned} & \|C_T(g(Y_1), \tau)\|_{L_p} \\ & \leq c_{(4)} c(A.1) \left\| \left( \sum_{i=1}^n \int_{\rho_{i-1}}^{\rho_i} (\rho_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \\ & \quad + c_{(4)} c(A.1) \delta \left\| \left( \int_0^T \int_0^t \left| \sum_{i=1}^n \chi_{\{\rho_{i-1} < u \leq t \leq \rho_i\}} \nu_u^{\rho_{i-1}} \right|^2 du dt \right)^{\frac{1}{2}} \right\|_{L_p} \\ & \leq c_{(4)} c(A.1) \left\| \left( \sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \end{aligned}$$

$$+c_{(4)}c_{(A.1)}\delta\left\|\left(\int_0^T\int_0^t\sum_{i=1}^n\chi_{(\rho_{i-1},\rho_i]}(u)|\nu_u^{\rho_{i-1}}|^2dudt\right)^{\frac{1}{2}}\right\|_{L_p}.$$

Because  $2 \leq p < \infty$ , we can continue by

$$\begin{aligned} & \left\|\left(\int_0^T\int_0^t\sum_{i=1}^n\chi_{(\rho_{i-1},\rho_i]}(u)|\nu_u^{\rho_{i-1}}|^2dudt\right)^{\frac{1}{2}}\right\|_{L_p} \\ & \leq \left(\int_0^T\left\|\left(\int_0^t\sum_{i=1}^n\chi_{(\rho_{i-1},\rho_i]}(u)|\nu_u^{\rho_{i-1}}|^2du\right)^{\frac{1}{2}}\right\|_{L_p}^2dt\right)^{\frac{1}{2}} \\ & \leq c_{(4)}\left(\int_0^T\left\|\int_0^t\sum_{i=1}^n\chi_{(\rho_{i-1},\rho_i]}(u)\nu_u^{\rho_{i-1}}dW_u\right\|_{L_p}^2dt\right)^{\frac{1}{2}} \\ & = c_{(4)}\left(\int_0^T\left\|C_t(g(Y_1),\tau)\right\|_{L_p}^2dt\right)^{\frac{1}{2}}, \end{aligned}$$

where we applied the Burkholder-Davies-Gundy inequalities. Combining these estimates we achieve

$$\begin{aligned} \|C_T(g(Y_1),\tau)\|_{L_p}^2 & \leq 2c_{(4)}^2c_{(A.1)}^2\left\|\left(\sum_{i=1}^n\int_{\tau_{i-1}}^{\tau_i}(\tau_i-t)H_G^2(t,Y_t)dt\right)^{\frac{1}{2}}\right\|_{L_p}^2 \\ & \quad + 2c_{(4)}^4c_{(A.1)}^2\int_0^T\left\|C_t(g(Y_1),\tau)\right\|_{L_p}^2dt. \end{aligned}$$

Gronwall's lemma thus implies that

$$\begin{aligned} & \|C_T(g(Y_1),\tau)\|_{L_p} \\ & \leq \sqrt{2}c_{(4)}c_{(A.1)}e^{c_{(4)}^4c_{(A.1)}^2}\left\|\left(\sum_{i=1}^n\int_{\tau_{i-1}}^{\tau_i}(\tau_i-t)H_G^2(t,Y_t)dt\right)^{\frac{1}{2}}\right\|_{L_p}. \end{aligned} \quad (15)$$

Finally, by  $T \uparrow 1$  we obtain the upper bound in Theorem 5.1.  $\square$

**Remark 5.3.** Our proof requires the assumption that the stopping time  $\tau_i$  is  $\mathcal{F}_{\tau_{i-1}}$ -measurable so that  $\rho_i$  is  $\mathcal{F}_{\rho_{i-1}}$ -measurable. For example we need that the field  $(\mu^\rho(t,u))_{t,u \in [0,1]}$  has the property that  $\mu^\rho(t,u)$  is  $\mathcal{F}_u$ -measurable. Moreover, in step (b) we used  $\mathbb{E}_{\mathcal{F}_{\rho_{i-1}}} \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2 dt = \int_{\rho_{i-1}}^{\rho_i} |m_{\rho_{i-1}}|^2 dt$ .

**Approximation with adapted time-nets in  $L_p$ .** We recall that the nets  $\tau_n^\theta$  are given by

$$t_{i,n}^\theta = 1 - \left(1 - \frac{i}{n}\right)^{\frac{1}{\theta}}.$$

The following result extends [13, Theorem 3.2] from the one-dimensional  $L_2$ -setting, but see also [19, Theorem 1] for a related  $d$ -dimensional  $L_2$ -result.

**Theorem 5.4.** *For  $2 \leq p < \infty$ ,  $0 < \theta \leq 1$  and  $g(Y_1) \in L_p$ , the following assertions are equivalent:*

- (i)  $\|D_1^{Y,\theta} g(Y_1)\|_p < \infty$ .
- (ii)  $\sup_{\tau \in \mathcal{T}^{\text{rand}}} \frac{\|C_1(g(Y_1), \tau)\|_p}{\|\sqrt{|\tau|_\theta}\|_\infty} < \infty$ .
- (iii)  $\sup_{n \geq 1} \sqrt{n} \|C_1(g(Y_1), \tau_n^\theta)\|_p < \infty$ .

In particular, for all  $\tau \in \mathcal{T}^{\text{rand}}$ ,

$$\|C_1(g(Y_1), \tau)\|_{L_p} \leq c_{(5.1)} \|\sqrt{|\tau|_\theta} D_1^{Y,\theta} g(Y_1)\|_{L_p}, \quad (16)$$

where  $c_{(5.1)} \geq 1$  is the constant from Theorem 5.1.

For the proof we need the following lemma that extends [13, Lemma 3.8].

**Lemma 5.5.** *Let  $0 < \theta \leq 1$  and  $0 < p < \infty$ . Assume that  $(\phi_t)_{t \in [0,1]}$  is a measurable process where all paths are continuous and non-negative. Then the following assertions are equivalent:*

- (i) *There exists a constant  $c_1 > 0$  such that*

$$\left\| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - u) \phi_u du \right\|_{L_p} \leq c_1 \sup_{1 \leq i \leq n} \frac{t_i - t_{i-1}}{(1 - t_{i-1})^{1-\theta}}$$

*for all deterministic time-nets  $0 = t_0 < t_1 < \dots < t_n = 1$ .*

- (ii) *There exists a constant  $c_2 > 0$  such that, for all  $n = 1, 2, \dots$ ,*

$$\left\| \sum_{i=1}^n \int_{t_{i-1,n}^\theta}^{t_{i,n}^\theta} (t_{i,n}^\theta - u) \phi_u du \right\|_{L_p} \leq \frac{c_2}{n}.$$

(iii) *There exists a constant  $c_3 > 0$  such that*

$$\left\| \int_0^1 (1-u)^{1-\theta} \phi_u du \right\|_{L_p} \leq c_3.$$

*Proof.* The implications (iii)  $\implies$  (i)  $\implies$  (ii) are similar to [13, Lemma 3.8]. For (ii)  $\implies$  (iii), take a sequence of deterministic nets  $\tau_n = (t_i^n)_{i=0}^n$  with  $0 = t_0^n < t_1^n < \dots < t_n^n = 1$  such that

$$|\tau_n| \leq \frac{\alpha}{n} \quad \text{and} \quad \sup_{1 \leq i \leq n} \frac{(1-t_{i-1}^n)^{1-\theta}}{t_i^n - t_{i-1}^n} \leq \beta n$$

for some  $\alpha, \beta > 0$  independent from  $n$  (see, for example, (10) and (12)). For a fixed  $0 < T < 1$  we define

$$N_T^n := \{i \in \{1, \dots, n\} : t_{i-1}^n < T\}$$

and observe that

$$\int_0^T (1-u)^{1-\theta} \phi_u du \leq \liminf_{n \rightarrow \infty} \sum_{i \in N_T^n} (1-t_{i-1}^n)^{1-\theta} \phi_{t_{i-1}^n} (t_i^n - t_{i-1}^n)$$

for all  $\omega \in \Omega$  because  $\phi$  is continuous on  $[0, T]$ . Hence

$$\begin{aligned} & \left\| \int_0^T (1-u)^{1-\theta} \phi_u du \right\|_{L_p} \\ & \leq \left\| \liminf_{n \rightarrow \infty} \left[ \sum_{i \in N_T^n} (1-t_{i-1}^n)^{1-\theta} \phi_{t_{i-1}^n} (t_i^n - t_{i-1}^n) \right] \right\|_{L_p} \\ & \leq \left\| \liminf_{n \rightarrow \infty} \left[ \sup_{1 \leq i \leq n} \frac{(1-t_{i-1}^n)^{1-\theta}}{t_i^n - t_{i-1}^n} \right] \left[ \sum_{i \in N_T^n} (t_i^n - t_{i-1}^n)^2 \phi_{t_{i-1}^n} \right] \right\|_{L_p} \\ & \leq \beta \left\| \liminf_{n \rightarrow \infty} n \left[ \sum_{i \in N_T^n} (t_i^n - t_{i-1}^n)^2 \phi_{t_{i-1}^n} \right] \right\|_{L_p}. \end{aligned}$$

Noticing that  $(t_i^n - t_{i-1}^n)^2 = 2 \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) du$  we continue with

$$\beta \left\| \liminf_{n \rightarrow \infty} n \left[ 2 \sum_{i \in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) du \phi_{t_{i-1}^n} \right] \right\|_{L_p}$$

$$\begin{aligned}
&\leq \beta \left\| \liminf_{n \rightarrow \infty} n \left[ 2 \sum_{i \in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \phi_u du \right. \right. \\
&\quad \left. \left. + \sum_{i \in N_T^n} \sup_{t_{i-1}^n \leq u < t_i^n} |\phi_u - \phi_{t_{i-1}^n}| (t_i^n - t_{i-1}^n)^2 \right] \right\|_{L_p} \\
&\leq \beta \left\| \liminf_{n \rightarrow \infty} n \left[ 2 \sum_{i \in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \phi_u du \right. \right. \\
&\quad \left. \left. + \sup_{i \in N_T^n} \sup_{t_{i-1}^n \leq u < t_i^n} |\phi_u - \phi_{t_{i-1}^n}| (t_i^n - t_{i-1}^n) \right] \right\|_{L_p} \\
&\leq \beta \left\| \liminf_{n \rightarrow \infty} \left[ 2n \sum_{i \in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \phi_u du \right. \right. \\
&\quad \left. \left. + \alpha \sup_{i \in N_T^n} \sup_{t_{i-1}^n \leq u < t_i^n} |\phi_u - \phi_{t_{i-1}^n}| \right] \right\|_{L_p} \\
&\leq \beta \left\| 2 \liminf_{n \rightarrow \infty} n \sum_{i \in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \phi_u du \right. \\
&\quad \left. + \alpha \limsup_{n \rightarrow \infty} \sup_{i \in N_T^n} \sup_{t_{i-1}^n \leq u < t_i^n} |\phi_u - \phi_{t_{i-1}^n}| \right\|_{L_p} \\
&= 2\beta \left\| \liminf_{n \rightarrow \infty} n \sum_{i \in N_T^n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \phi_u du \right\|_{L_p} \\
&\leq 2\beta \liminf_{n \rightarrow \infty} n \left\| \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \phi_u du \right\|_{L_p},
\end{aligned}$$

where we used Fatou's lemma. Finally, by monotone convergence this implies that

$$\left\| \int_0^1 (1-u)^{1-\theta} \phi_u du \right\|_{L_p} \leq 2\beta \liminf_{n \rightarrow \infty} n \left\| \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i^n - u) \phi_u du \right\|_{L_p}. \quad \square$$

*Proof of Theorem 5.4.* First, we employ Theorem 5.1 to confirm equation (16) by

$$\|C_1(g(Y_1), \tau)\|_{L_p} \leq c_{(5.1)} \left\| \left( \sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p}$$

$$\leq c_{(5.1)} \left\| \sqrt{|\tau|_\theta} \left( \sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (1-t)^{1-\theta} H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p}.$$

Part (i) $\implies$  (ii) follows from (16) and part (ii) $\implies$  (iii) from  $|\tau_n^\theta|_\theta \leq \frac{1}{\theta n}$  (see (11)). To show that (iii) $\implies$  (i) we apply Theorem 5.1 and (13) to see that

$$\begin{aligned} \frac{c}{\sqrt{n}} &\geq \|C_1(g(Y_1), \tau_n^\theta)\|_{L_p} \\ &\geq \frac{1}{c_{(5.1)}} \left\| \left( \sum_{i=1}^n \int_{t_{i-1,n}^\theta}^{t_{i,n}^\theta} (t_{i,n}^\theta - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p}. \end{aligned}$$

Lemma 5.5 completes the proof.  $\square$

**Approximation with equidistant time-nets in  $L_p$ .** Here we extend [7, Theorem 2.3] and [13, Theorem 3.5 for  $q = \infty$ ] to the  $L_p$ -case and [23, Theorem 1.2].

**Theorem 5.6.** *For  $2 \leq p < \infty$ ,  $0 < \theta < 1$  and  $g(Y_1) \in L_p$  the following assertions are equivalent:*

- (i)  $f \in \mathbb{B}_{p,\infty}^\theta$ .
- (ii)  $\sup_{\tau \in \mathcal{T}^{\text{rand}}} \frac{\|C_1(g(Y_1); \tau)\|_p}{\|\tau\|_\infty^{\frac{\theta}{2}}} < \infty$ .
- (iii)  $\sup_{n=1,2,\dots} n^{\frac{\theta}{2}} \|C_1(g(Y_1); \tau_n)\|_p < \infty$ , where  $\tau_n = (i/n)_{i=0}^n$  are the equidistant time-nets.

In particular, for  $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$  with  $p \leq q, r \leq \infty$  and for all  $\tau \in \mathcal{T}^{\text{rand}}$ ,

$$\begin{aligned} &\|C_1(g(Y_1), \tau)\|_p \\ &\leq c_{(5.1)} \left( \int_0^1 \|\sqrt{\psi(t)}\|_q^2 (1-t)^{\theta-2} dt \right)^{\frac{1}{2}} \sup_{t \in [0,1]} (1-t)^{1-\frac{\theta}{2}} \|H_G(t, Y_t)\|_r, \quad (17) \end{aligned}$$

where  $c_{(5.1)} \geq 1$  is the constant from Theorem 5.1 and

$$\psi(t, \omega) := \left( \max_{i=1,\dots,n} |\tau_i(\omega) - \tau_{i-1}(\omega)| \right) \wedge (1-t).$$



**Remark 5.7.** The order for the equidistant nets can also be obtained from Theorem 5.4 under the condition  $\|D_1^{Y,\theta}g(Y_1)\|_p < \infty$  because  $|(i/n)_{i=0}^n|_\theta = n^{-\theta}$ .

*Proof of Theorem 5.6.* To verify (17) we use Theorem 5.1 and derive that

$$\begin{aligned}
\|C_1(g(Y_1), \tau)\|_p &\leq c_{(5.1)} \left\| \left( \sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} (\tau_i - t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_p \\
&\leq c_{(5.1)} \left\| \left( \int_0^1 \psi(t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_p \\
&\leq c_{(5.1)} \left( \int_0^1 \|\sqrt{\psi(t)} H_G(t, Y_t)\|_p^2 dt \right)^{\frac{1}{2}} \\
&\leq c_{(5.1)} \left( \int_0^1 \|\sqrt{\psi(t)}\|_q^2 \|H_G(t, Y_t)\|_r^2 dt \right)^{\frac{1}{2}} \\
&\leq c_{(5.1)} \left( \int_0^1 \|\sqrt{\psi(t)}\|_q^2 (1-t)^{\theta-2} dt \right)^{\frac{1}{2}} \\
&\quad \sup_{t \in [0,1)} (1-t)^{1-\frac{\theta}{2}} \|H_G(t, Y_t)\|_r.
\end{aligned}$$

Part (i)  $\implies$  (ii): We first observe that for

$$|\tau|(\omega) = \max_{i=1, \dots, n} |\tau_i(\omega) - \tau_{i-1}(\omega)|,$$

we can compute (for  $q = \infty$  and  $r = p$ )

$$\begin{aligned}
&\int_0^1 \|\sqrt{\psi(t)}\|_\infty^2 (1-t)^{\theta-2} dt \\
&= \|\tau\|_\infty \int_0^{1-\|\tau\|_\infty} (1-t)^{\theta-2} dt + \int_{1-\|\tau\|_\infty}^1 (1-t)^{\theta-1} dt \\
&= \|\tau\|_\infty \frac{1}{1-\theta} (\|\tau\|_\infty^{\theta-1} - 1) + \frac{1}{\theta} \|\tau\|_\infty^\theta \\
&\leq \frac{1}{\theta(1-\theta)} \|\tau\|_\infty^\theta,
\end{aligned}$$

so that letting  $q = \infty$  in (17) we obtain

$$\|C_1(g(Y_1), \tau)\|_p \leq \frac{c_{(5.1)}}{\sqrt{\theta(1-\theta)}} \|\tau\|_\infty^{\frac{\theta}{2}} \sup_{t \in [0,1)} (1-t)^{1-\frac{\theta}{2}} \|H_G(t, Y_t)\|_p.$$

It remains to check that

$$\sup_{t \in [0,1)} (1-t)^{1-\frac{\theta}{2}} \|H_G(t, Y_t)\|_p < \infty$$

whenever  $f \in \mathbb{B}_{p,\infty}^\theta$ . This follows from Theorem 3.1, where we additionally use (9) and the a-priori estimate

$$\sup_{t \in [0,1)} (1-t)^{\frac{1}{2}} \|\nabla F(t, W_t)\|_{L_p} < \infty \quad (18)$$

from Lemma A.3 in the case  $Y$  is the geometric Brownian motion.

The implication (ii)  $\implies$  (iii) is trivial.

Part (iii)  $\implies$  (i): Employing Theorem 5.1 and (13) we achieve

$$\begin{aligned} cn^{-\frac{\theta}{2}} &\geq \|C_1(g(Y_1), \tau_n)\|_{L_p} \\ &\geq \frac{1}{c_{(5.1)}} \left\| \left( \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left( \frac{i}{n} - t \right) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\geq \frac{1}{c_{(5.1)}} \left\| \left( \int_{\frac{n-1}{n}}^1 (1-t) H_G^2(t, Y_t) dt \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\geq \frac{1}{c_{(5.1)}} \left\| \left( \int_{\frac{n-1}{n}}^1 (1-t) H_G^2 \left( 1 - \frac{1}{n}, Y_{1-\frac{1}{n}} \right) dt \right)^{\frac{1}{2}} \right\|_{L_p} \\ &= \frac{1}{c_{(5.1)}} \sqrt{\frac{1}{2} \frac{1}{n}} \left\| H_G \left( 1 - \frac{1}{n}, Y_{1-\frac{1}{n}} \right) \right\|_{L_p}, \end{aligned}$$

where we use in the last inequality the martingale property of the processes  $\left( \left( \sigma_{kk} \sigma_{ll} \frac{\partial^2 G}{\partial y_k \partial y_l} \right) (t, Y_t) \right)_{t \in [0,1)}$  and properties of conditional expectations of vector valued random variables. The estimate above means that

$$\left\| H_G \left( 1 - \frac{1}{n}, Y_{1-\frac{1}{n}} \right) \right\|_{L_p} \leq \sqrt{2} c c_{(5.1)} n^{1-\frac{\theta}{2}}$$

for all  $n = 2, 3, \dots$ . Consequently,

$$\|H_G(t, Y_t)\|_{L_p} \leq 2^{1-\frac{\theta}{2}} \sqrt{2} c c_{(5.1)} (1-t)^{\frac{\theta}{2}-1},$$

which follows from the monotonicity of  $\|H_G(t, Y_t)\|_{L_p}$ . Theorem 3.1 completes the proof, where we use (18) again.  $\square$

## A Appendix

A key step in the proof of Theorem 5.1 is the following known formulation of the Burkholder-Davis-Gundy inequalities.

**Lemma A.1.** *Assume that  $\mu : [0, 1] \times [0, 1] \times \Omega \rightarrow \mathbb{R}^{d \times d}$  satisfies the following assumptions:*

- (i)  $\mu : [0, 1] \times [0, u] \times \Omega \rightarrow \mathbb{R}^{d \times d}$  is  $\mathcal{B}([0, 1]) \times \mathcal{B}([0, u]) \times \mathcal{F}_u$ -measurable for all  $u \in [0, 1]$ .
- (ii)  $\int_0^1 \int_0^1 \mathbb{E} |\mu(t, u)|^2 du dt < \infty$ , where  $\int_0^1 \mathbb{E} |\mu(t, u)|^2 du < \infty$  for all  $t \in [0, 1]$ .
- (iii)  $(\int_0^1 \mu(t, u) dW_u)_{t \in [0, 1]}$  is a measurable modification.

Then, for  $1 < p < \infty$ , there exists a constant  $c_{(A.1)} \geq 1$  depending only on  $p$  such that

$$\left\| \left( \int_0^1 \left| \int_0^1 \mu(t, u) dW_u \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p} \sim_{c_{(A.1)}} \left\| \left( \int_0^1 \int_0^1 |\mu(t, u)|^2 du dt \right)^{\frac{1}{2}} \right\|_{L_p}.$$

*Proof.* For the convenience of the reader we sketch the proof. By a further modification we can assume that  $((\int_0^1 \mu(t, u) dW_u)(\omega))_{t \in [0, 1]} \in L_2[0, 1]$  for all  $\omega \in \Omega$  because of assumption (ii). Assume that  $(h_n)_{n=0}^\infty$  is the orthonormal basis of Haar-functions in  $L_2[0, 1]$  and that  $\mu_k(t, u)$  is the  $k$ -th row of  $\mu(t, u)$ . Letting

$$L_u^{n,k} := \int_0^1 h_n(t) \mu_k(t, u) dt$$

and using a stochastic Fubini argument we get that

$$\left\| \left( \int_0^1 \left| \int_0^1 \mu(t, u) dW_u \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p} = \left\| \left( \sum_{n=0}^\infty \sum_{k=1}^d \left| \int_0^1 L_u^{n,k} dW_u \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p}.$$

Using the Burkholder-Davis-Gundy inequalities (4) we obtain that

$$\left\| \left( \int_0^1 \left| \int_0^1 \mu(t, u) dW_u \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L_p}$$

$$\begin{aligned}
& \sim_{c_{(4)}} \left\| \left( \sum_{n=0}^{\infty} \sum_{k=1}^d \int_0^1 |L_u^{n,k}|^2 du \right)^{\frac{1}{2}} \right\|_{L_p} \\
& = \left\| \left( \sum_{k=1}^d \int_0^1 \int_0^1 |\mu_k(t, u)|^2 dt du \right)^{\frac{1}{2}} \right\|_{L_p} \\
& = \left\| \left( \int_0^1 \int_0^1 |\mu(t, u)|^2 dt du \right)^{\frac{1}{2}} \right\|_{L_p}.
\end{aligned}$$

□

**Lemma A.2.** *Let  $1 < p < \infty$ ,  $g(Y_1) \in L_p$  and  $0 < t < 1$ , and let  $a = (a_1, \dots, a_d)$  be a multi-index of differentiation. Assume that  $G$  is given by (1). Then*

$$\left\| \sup_{0 \leq s \leq t} |D_y^a G(s, Y_s)| \right\|_{L_q} < \infty \quad \text{for } 0 < q < q(p, t) := \frac{p-1+t}{t}.$$

*Sketch of the proof.* We use the notation (7) and (8) and consider first the case that  $Y$  is the Brownian motion. A simple direct computation gives the hyper-contraction property

$$|D_x^a F(t, x)| \leq C(q, t, a) \|f\|_{L_p(\mathbb{R}^d, \gamma_d)} e^{\frac{|x|^2}{2tq}}$$

for  $0 < t < 1$  and  $0 < q < q(p, t)$ . Moreover,

$$D_x^a F(s, x) = \mathbb{E} D_x^a F(t, x + W_{t-s})$$

for  $0 \leq s \leq t < 1$  directly implies that  $(D_s^a F(s, W_s))_{s \in [0, t]}$  is an  $L_q$ -martingale. Therefore we can exploit Doob's maximal inequality for  $1 < q < q(p, t)$  to conclude

$$\mathbb{E} \sup_{0 \leq s \leq t} |D_x^a F(s, W_s)|^q < \infty \quad \text{for all } 0 < q < q(p, t). \quad (19)$$

The case of the geometric Brownian motion can be deduced from the case of the Brownian motion. Using the notation (7) and (8) to switch between the Brownian and geometric Brownian motion, we get for  $0 \leq t < 1$  that

$$D_y^a G(t, Y_t) = \left[ \prod_{k=1}^d (Y_t^k)^{-a_k} \right] \sum_{0 \leq b \leq a} \kappa_a^b D_x^b F(t, W_t)$$

where  $0 \leq b \leq a$  is the coordinate-wise ordering and  $\kappa_a^b$  are fixed coefficients. Using (19), the integrability properties of the geometric Brownian motion and Hölder's inequality, we conclude that

$$\mathbb{E} \sup_{0 \leq s \leq t} |D_y^a G(s, Y_s)|^q < \infty \quad \text{for all } 0 < q < q(p, t). \quad \square$$

The following estimates are known for more general processes than the Brownian motion (see [15] and [9, Remark 3]). In our case they can be easily verified by using the martingale property of the processes  $(\nabla F(t, W_t))_{t \in [0,1]}$  and  $(D^2 F(t, W_t))_{t \in [0,1]}$ .

**Lemma A.3.** *Let  $2 \leq p < \infty$ . Assume that  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable with  $f \in L_p(\mathbb{R}^d, \gamma_d)$  and that  $F : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is given by  $F(t, x) := \mathbb{E}f(x + W_{1-t})$ . Then there exists a constant  $c_{(A.3)} > 0$  depending only on  $p$  such that, for all  $0 \leq t < 1$ ,*

- (i)  $\|\nabla F(t, W_t)\|_{L_p} \leq c_{(A.3)}(1-t)^{-\frac{1}{2}} \|f(W_1) - F(t, W_t)\|_{L_p},$
- (ii)  $\|D^2 F(t, W_t)\|_{L_p} \leq c_{(A.3)}(1-t)^{-1} \|f(W_1) - F(t, W_t)\|_{L_p}.$

Next we state some Hardy type inequalities we have used in the paper.

**Proposition A.4.** *Let  $0 < \theta < 1$ ,  $2 \leq q \leq \infty$  and let  $d^k : [0, 1] \rightarrow [0, \infty)$ ,  $k = 0, 1, 2$ , be measurable functions. Assume that*

$$\frac{1}{\alpha}(1-t)^{\frac{k}{2}} d^k(t) \leq d^0(t) \leq \alpha \left( \int_t^1 [d^1(s)]^2 ds \right)^{\frac{1}{2}} \quad \text{for } t \in [0, 1]$$

and for  $k = 1, 2$ , and that

$$d^1(t) \leq A + \alpha \left( \int_0^t [d^2(u)]^2 du \right)^{\frac{1}{2}} \quad \text{for } t \in [0, 1]$$

for some  $A \geq 0$  and  $\alpha > 0$ . Then

$$\left\| (1-t)^{-\frac{\theta}{2}} d^0(t) \right\|_{L_q([0,1], \frac{dt}{1-t})} \sim_{c_{(A.4)}} \left\| (1-t)^{\frac{1-\theta}{2}} d^1(t) \right\|_{L_q([0,1], \frac{dt}{1-t})}$$

and

$$\begin{aligned}
& \left\| (1-t)^{\frac{2-\theta}{2}} d^2(t) \right\|_{L_q([0,1), \frac{dt}{1-t})} \\
& \leq c_{(A.4)} \left\| (1-t)^{-\frac{\theta}{2}} d^0(t) \right\|_{L_q([0,1), \frac{dt}{1-t})} \\
& \leq c_{(A.4)}^2 \left[ A + \left\| (1-t)^{\frac{2-\theta}{2}} d^2(t) \right\|_{L_q([0,1), \frac{dt}{1-t})} \right],
\end{aligned}$$

where  $c_{(A.4)} \geq 1$  depends at most on  $(\alpha, \theta, q)$ . If the functions  $d^1$  and  $d^2$  are non-decreasing, then the inequalities are true for  $1 \leq q < 2$  as well.

From Proposition A.4 it follows that

$$A + \left\| (1-t)^{\frac{k-\theta}{2}} d^k(t) \right\|_{L_q} \sim_{c(20)} A + \left\| (1-t)^{\frac{l-\theta}{2}} d^l(t) \right\|_{L_q}. \quad (20)$$

for  $L_q = L_q([0,1), \frac{dt}{1-t})$ ,  $k, l = 0, 1, 2$  and  $c(20) := [1 + c_{(A.4)}]^2$ . To prove Proposition A.4 we need

**Lemma A.5.** *Let  $0 < \theta < 1$ ,  $2 \leq q \leq \infty$  and let  $\phi : [0, 1) \rightarrow [0, \infty)$  be a measurable function. Then there is a constant  $c_{(A.5)} > 0$ , depending at most on  $\theta$ , such that*

$$\begin{aligned}
& \left\| (1-t)^{\frac{1-\theta}{2}} \left( \int_0^t \phi(u)^2 du \right)^{\frac{1}{2}} \right\|_{L_q([0,1), \frac{dt}{1-t})} \\
& \leq c_{(A.5)} \left\| (1-t)^{1-\frac{\theta}{2}} \phi(t) \right\|_{L_q([0,1), \frac{dt}{1-t})}. \quad (21)
\end{aligned}$$

Moreover, if  $\phi$  is non-decreasing, the inequality is true for  $1 \leq q < 2$  as well.

*Proof.* (a) For  $2 \leq q \leq \infty$ , we can use Hardy's inequality (see e.g. [1, Theorem 3.3.9]): for  $-\infty < \lambda < 1$  and  $1 \leq r < \infty$ , and a measurable  $\psi : (0, \infty) \rightarrow [0, \infty)$ ,

$$\left( \int_0^\infty \left[ t^{1-\lambda} \int_t^\infty \psi(s) \frac{ds}{s} \right]^r \frac{dt}{t} \right)^{\frac{1}{r}} \leq \frac{1}{1-\lambda} \left( \int_0^\infty [t^{1-\lambda} \psi(t)]^r \frac{dt}{t} \right)^{\frac{1}{r}}$$

and the same with the supremum norm if  $r = \infty$ . With the notation  $r := \frac{q}{2}$ ,  $g(t) = [\phi(t)]^2$ , and  $\lambda = \theta$ , we compute, in the case  $2 \leq q < \infty$ ,

$$\left\| (1-t)^{\frac{1-\theta}{2}} \left( \int_0^t \phi(u)^2 du \right)^{\frac{1}{2}} \right\|_{L_q([0,1), \frac{dt}{1-t})}^2$$

$$\begin{aligned}
&= \left( \int_0^1 \left[ (1-t)^{1-\theta} \int_0^t g(u) du \right]^r \frac{dt}{1-t} \right)^{\frac{1}{r}} \\
&= \left( \int_0^\infty \left[ s^{1-\theta} \int_s^\infty h(v) dv \right]^r \frac{ds}{s} \right)^{\frac{1}{r}},
\end{aligned}$$

where  $h(v) = g(1-v)\chi_{(0,1]}(v)$ . Now we use Hardy's inequality for  $\psi(v) = vh(v)$  and continue with

$$\begin{aligned}
&\left( \int_0^\infty \left[ s^{1-\theta} \int_s^\infty \psi(v) \frac{dv}{v} \right]^r \frac{ds}{s} \right)^{\frac{1}{r}} \\
&\leq \frac{1}{1-\theta} \left( \int_0^\infty [s^{1-\theta} \psi(s)]^r \frac{ds}{s} \right)^{\frac{1}{r}} \\
&= \frac{1}{1-\theta} \left( \int_0^\infty [s^{2-\theta} h(s)]^r \frac{ds}{s} \right)^{\frac{1}{r}} \\
&= \frac{1}{1-\theta} \left\| (1-t)^{1-\frac{\theta}{2}} \phi(t) \right\|_{L_q([0,1), \frac{dt}{1-t})}^2,
\end{aligned}$$

and the proof is complete for  $2 \leq q < \infty$ . The case  $q = \infty$  is analogous.

(b) For  $1 \leq q < 2$  we use a different argument. First, we define  $r := \frac{2}{q}$  so that  $1 < r \leq 2$ . For  $0 < T < 1$  we compute

$$\begin{aligned}
\int_0^1 (1-t)^{\frac{1-\theta}{r}} \left( \int_0^t \chi_{[T,1)}(u) du \right)^{\frac{1}{r}} \frac{dt}{1-t} &= \int_0^1 (1-t)^{\frac{1-\theta}{r}} (t-T)_+^{\frac{1}{r}} \frac{dt}{1-t} \\
&\leq (1-T)^{\frac{1}{r}} \int_T^1 (1-t)^{\frac{1-\theta}{r}-1} dt \\
&= c \int_T^1 (1-t)^{\frac{2-\theta}{r}} \chi_{[T,1)}(t) \frac{dt}{1-t}
\end{aligned}$$

with  $c := \frac{2-\theta}{1-\theta}$ . This proves the desired inequality for  $\psi^{(T)}(t) := \chi_{[T,1)}(t)$ . Next, we define  $\psi := \phi^q$  so that  $\psi^r = \phi^2$ . By assumption,  $\phi$  is non-decreasing, and so is  $\psi$ , too. Now, we can approximate  $\psi$  from below by a sum of functions like  $\psi^{(T)}$ : for each integer  $n \geq 1$ , we find  $\alpha_k^n \geq 0$ ,  $k = 0, \dots, 2^n - 1$  and  $0 = t_0^n < t_1^n < \dots < t_{2^n-1}^n < t_{2^n}^n = 1$  such that

$$\psi_n(t) := \sum_{k=0}^{2^n-1} \alpha_k^n \psi^{(t_k^n)}(t) \rightarrow \psi(t)$$

for almost all  $t \in [0, 1)$  and  $\psi_{n-1} \leq \psi_n$  for all  $n \geq 2$ . Then, since  $r \geq 1$ ,

$$\begin{aligned}
& \int_0^1 (1-t)^{\frac{1-\theta}{r}} \left( \int_0^t \psi_n(u)^r du \right)^{\frac{1}{r}} \frac{dt}{1-t} \\
& \leq \int_0^1 (1-t)^{\frac{1-\theta}{r}} \sum_{k=0}^{2^n-1} \alpha_k^n \left( \int_0^t \psi^{(t_k^n)}(u) du \right)^{\frac{1}{r}} \frac{dt}{1-t} \\
& \leq \sum_{k=0}^{2^n-1} \alpha_k^n c \int_0^1 (1-t)^{\frac{2-\theta}{r}} \psi^{(t_k^n)}(t) \frac{dt}{1-t} \\
& = c \int_0^1 (1-t)^{\frac{2-\theta}{r}} \psi_n(t) \frac{dt}{1-t}
\end{aligned}$$

and the claim follows by monotone convergence.  $\square$

*Proof of Proposition A.4.* (a) Our assumptions imply for all  $1 \leq q \leq \infty$  that

$$\left\| (1-t)^{\frac{1-\theta}{2}} d^1(t) \right\|_{L_q([0,1), \frac{dt}{1-t})} \leq \alpha \left\| (1-t)^{-\frac{\theta}{2}} d^0(t) \right\|_{L_q([0,1), \frac{dt}{1-t})}$$

and

$$\left\| (1-t)^{\frac{2-\theta}{2}} d^2(t) \right\|_{L_q([0,1), \frac{dt}{1-t})} \leq \alpha \left\| (1-t)^{-\frac{\theta}{2}} d^0(t) \right\|_{L_q([0,1), \frac{dt}{1-t})}.$$

(b) Next we observe that

$$\begin{aligned}
& \left\| (1-t)^{-\frac{\theta}{2}} d^0(t) \right\|_{L_q([0,1), \frac{dt}{1-t})} \\
& \leq \alpha \left\| (1-t)^{-\frac{\theta}{2}} \left( \int_t^1 [d^1(s)]^2 ds \right)^{\frac{1}{2}} \right\|_{L_q([0,1), \frac{dt}{1-t})} \\
& = \alpha \left\| (1-t)^{\frac{1-\theta}{2}} \left( \frac{1}{1-t} \int_t^1 [d^1(s)]^2 ds \right)^{\frac{1}{2}} \right\|_{L_q([0,1), \frac{dt}{1-t})} \\
& \leq \alpha \theta^{-\max\{\frac{1}{2}, \frac{1}{q}\}} \left\| (1-t)^{\frac{1-\theta}{2}} d^1(t) \right\|_{L_q([0,1), \frac{dt}{1-t})},
\end{aligned}$$

where we used [13, Formula (14)] (the condition that  $\psi$  in [13] is continuous in the case  $1 \leq q < 2$  is not necessary).



(c) To prove the remaining inequality we continue from (b) with Lemma A.5 to

$$\begin{aligned}
& \left\| (1-t)^{\frac{1-\theta}{2}} d^1(t) \right\|_{L_q([0,1), \frac{dt}{1-t})} \\
& \leq \left\| (1-t)^{\frac{1-\theta}{2}} \left[ A + \alpha \left( \int_0^t d^2(u) du \right)^{\frac{1}{2}} \right] \right\|_{L_q([0,1), \frac{dt}{1-t})} \\
& \leq A \left\| (1-t)^{\frac{1-\theta}{2}} \right\|_{L_q([0,1), \frac{dt}{1-t})} + \alpha c(A.5) \left\| (1-t)^{1-\frac{\theta}{2}} d^2(t) \right\|_{L_q([0,1), \frac{dt}{1-t})}.
\end{aligned}$$

□

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